The K-level widths increase rapidly with atomic number beyond iodine, up to 96 eV for uranium. It seems unlikely that K-edge resonances for any of these heavy elements will be strong enough to compensate for the spreading of effects over wavelength any better than in the present case. No L absorption edges occur in existing elements above 33 keV. Thus the high-energy region is not favorable for applications which would exploit this kind of dichroism or birefringence. Those who seek to avoid these complications may welcome this result. Another conclusion is that the search for examples of greatest anisotropy should be directed toward the long-wavelength region where levels are narrow.

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References


**A Memory-Efficient Fast Fourier Transformation Algorithm for Crystallographic Refinement on Supercomputers**

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**Abstract**

A vectorizable algorithm for fast computation of structure factors and derivatives during refinement of macromolecular structures is presented. It is based on fast Fourier transformations on subgrids that cover the unit cell of the crystal. The use of subgrids allows reduction of the total memory requirements for the computations without producing large overheads. The algorithm is applicable to all space groups. The
performance of the algorithm on a conventional scalar computer as well as on supercomputers is discussed.

Introduction

The step in crystallographic refinement that consumes the most central processor unit (CPU) time is the evaluation of the crystallographic residual and its first derivatives with respect to the atomic coordinates of the molecule. This applies to restrained least-squares refinement (Sussmann, Holbrook, Church & Kim, 1977; Jack & Levitt, 1978; Konnert & Hendrickson, 1980; Moss & Morffew, 1982) as well as to refinement by simulated annealing (Brünger, Kuriyan & Karplus, 1987; Brünger, Karplus & Petsko, 1989; Brünger, 1988a, b). The CPU time needed to compute the structure factors through the direct summation method exceeds the CPU time for a single evaluation of the stereochemical restraints by up to two orders of magnitude. For example, in the case of aspartate aminotransferase, a protein composed of 396 amino acids, the evaluation of the crystallographic residual and its first derivatives through direct summation takes 103 s at 2.8 Å resolution whereas the evaluation of the stereochemical restraints takes only 2.8 s on a Cray-XMP. The efficient calculation of the crystallographic residual and its first derivatives is of particular concern for crystallographic refinement by simulated annealing since this method requires many more updates of the structure factors than restrained least-squares refinement.

Evaluation of the atomic electron density on a finite grid followed by fast Fourier transformation (FFT) provides a way to improve significantly the speed of the calculation of the crystallographic residual and its first derivatives (Cooley & Tukey, 1965; Ten Eyck, 1973, 1977; Agarwal, 1978, 1981; Isaacs, 1984; Tronrud, Ten Eyck & Matthews, 1987) concern the organization of the finite grid and the application of symmetry operators in reciprocal space. If the electron density grid is too large to fit into the available memory of the computer, the FFT is carried out on subgrids and the results are then accumulated. The subgrid can be chosen small enough to fit into the memory of the computer, thus optimizing the performance of the FFT.

It has already been pointed out (Isaacs, 1984) that the modelling of the electron density on the finite grid is an expensive part of the computation. To reduce the time spent in the calculation of the atomic electron density, it is computed only for a set of unique atoms. The FFT is carried out over the unit cell, and the transposed symmetry operators of the crystal space group are then applied to the computed structure factors. This yields the identical result to computing the electron density for all atoms in the unit cell.

The following sections describe the mathematical basis for the subgrid fast Fourier transformation (SGFFT) algorithm and its current implementation on supercomputers. In the last section the performance of the algorithm is discussed for three representative examples.

Notation

We define $\Omega$ as a unique set of observed reflections with indices $hkl$ that satisfy certain selection criteria. Usually, the reflections are selected based on their significance, i.e. whether the intensity is larger than a certain multiple of the standard deviation. The index triple $(h, k, l)$ is denoted as $h$. The space group of the crystal will be described in terms of $n_s$ symmetry operators $(\mathcal{O}_s, t_s)$ composed of the $3 \times 3$ matrix $\mathcal{O}_s$ representing a rotation and a vector $t_s$ representing a translation. It is assumed that the atomic model consists of $n_a$ unique (not symmetry-related) atoms. The set of unique atoms may coincide with a standard crystallographic asymmetric unit, but this is not a necessary condition for this work. The coordinates of every atom in the unit cell can then be generated by application of a symmetry operator to the coordinates of an atom in the set of unique atoms. The vector $r_i$ denotes the orthogonal coordinates of atom $i$ in Å. $\mathcal{F}$ is the $3 \times 3$ matrix that converts orthogonal Å coordinates into fractional coordinates; $\mathcal{F}^*$ denotes the transpose of it. The columns of $\mathcal{F}^*$ are equal to the reciprocal unit-cell vectors $a$, $b$, $c$. The length of
the reciprocal-lattice vector \( s = \mathcal{F}^{-1} h \) which corresponds to the reflection \( h \) is equal to \( 2 \sin \theta / \lambda \), where \( \lambda \) is the wavelength of the X-rays and \( \theta \) is the scattering angle.

**Target function**

Crystallographic refinement can be understood as a nonlinear optimization problem with the aim of finding a minimum close to the global minimum of a target function \( T \),

\[
T(p_1, p_2, \ldots, p_n) = \Delta[|F_{\text{obs}}|, \phi_{\text{obs}}, F_{\text{calc}}(p_1, p_2, \ldots, p_n)] + E(p_1, p_2, \ldots, p_n)
\]

where \( \Delta \) is a function of the observed structure-factor amplitudes \( |F_{\text{obs}}| \), observed phase information \( \phi_{\text{obs}} \), and the calculated structure factors of the atomic model \( F_{\text{calc}} \). \( E \) comprises stereochemical and other interactions of the macromolecule; the quantities \( p_1, p_2, \ldots, p_n \) represent the variables of the system, such as the atomic positions or individual atomic \( B \) factors. The FFT refinement programs of Agarwal (1978) and Tronrud, Ten Eyck & Matthews (1987) use steepest descent or conjugent gradient minimization (Fletcher & Reeves, 1964) to optimize \( T \). The recently developed method of crystallographic refinement by simulated annealing (Brünger, Kuriyan & Karplus, 1987; Brünger, Karplus & Petsko, 1989; Brünger, 1988a) uses molecular dynamics to overcome local minima. Molecular dynamics as well as conjugate gradient minimization require the repeated computation of \( T \) and its first derivatives with respect to the model parameters \( p_i \). Higher-order minimization methods require the computation of second derivatives as well. Since the computation of \( E(p_1, p_2, \ldots, p_n) \) and its first derivatives is straightforward (cf. Konnert & Hendrickson, 1980) we are only concerned here with the computation of \( \Delta \) and its first derivatives.

In crystallographic refinement the function \( \Delta \) usually consists of the crystallographic residual

\[
\sum_{h \in \Omega} W_h[|F_{\text{obs}}(h)| - k|F_{\text{calc}}(h)|]^2,
\]

which is a sum of the weighted differences between observed \( |F_{\text{obs}}| \) and calculated \( |F_{\text{calc}}| \) structure-factor amplitudes. The sum extends over the set of all observed reflections \( \Omega \). \( W_h \) are individual weights for each reflection \( h \). The scale factor \( k \) is set to the value which makes the derivative of the crystallographic residual with respect to \( k \) zero,

\[
k = \sum_{h \in \Omega} W_h|F_{\text{obs}}(h)||F_{\text{calc}}(h)| \left/ \left( \sum_{h \in \Omega} W_h|F_{\text{calc}}(h)|^2 \right) \right. .
\]

This is a necessary condition to make the crystallographic residual minimal.

It should be noted that the following derivations do not depend on the particular form of \( \Delta(|F_{\text{obs}}|, \phi_{\text{obs}}, F_{\text{calc}}) \). For example, phase information can be added to the crystallographic residual (cf. Brünger, 1988b).

**Calculated structure factors**

The structure factors of a crystal are given by Fourier transformation of the electron density,

\[
F_{\text{calc}}(h) = \int_{\mathbf{g} \in \mathcal{V}} \rho(\mathcal{F}^{-1}g) \exp(2\pi i h . g) \, dg.
\]

The vectors \( g \) are defined in fractional coordinate space. The integration is carried out over the standard unit cell \( \mathcal{V} = (0 \ldots 1, 0 \ldots 1, 0 \ldots 1) \). The argument \( r = \mathcal{F}^{-1}g \) of the electron density \( \rho \) is specified in orthogonal Å coordinates.

The electron density \( \rho \) is a superposition of the individual electron densities of all atoms in the unit cell of the crystal. Using the crystallographic symmetry operators one can write

\[
\rho(r) = \sum_{s=1}^{n_o} \rho^A(\mathcal{C}_s r + t_s)
\]

where the sum runs through all symmetry operators \( (\mathcal{C}_s, t_s) \) of the space group. The term \( \rho^A \) is the superposition of the electron densities \( \rho_i \) of a set of unique (i.e. not symmetry-related) atoms,

\[
\rho^A(r) = \sum_{i=1}^{n_a} \rho_i(r)
\]

where \( n_a \) is the number of unique atoms. Let \( F^A_{\text{calc}}(h) \) be the structure factors generated by this unique set of atoms,

\[
F^A_{\text{calc}}(h) = \int_{\mathbf{g} \in \mathcal{V}} \rho^A(\mathcal{F}^{-1}g) \exp(2\pi i h . g) \, dg.
\]

Using (5) and (4) one can derive the result

\[
F_{\text{calc}}(h) = \sum_{s=1}^{n_o} F^A_{\text{calc}}(\mathcal{C}_s^* h) \exp(2\pi i h . t_s)
\]

where \( \mathcal{C}_s^* \) denotes the transpose of \( \mathcal{C}_s \). The coefficient \( \exp(2\pi i h . t_s) \) arises from the translational part of the symmetry operator \( (\mathcal{C}_s, t_s) \). Thus, the structure factors \( F_{\text{calc}}(h) \) can be obtained by computing the structure factors of a unique set of atoms and then accumulating all symmetry-related structure factors.

One possibility for explicit computation of (4) is to carry out the Fourier transformation for each atom translated to the origin, which yields the familiar 'direct summation' formula,

\[
F_{\text{calc}}(h) = \sum_{i=1}^{n_a} f_i(h) \sum_{s=1}^{n_o} \exp\left[2\pi i h . (\mathcal{C}_s \mathcal{F} r_i + t_s)\right].
\]

This equation is valid for all space groups. More efficient expressions can be obtained for each space
The form factors $f_i(h)$ are Fourier transforms of the electron densities $\rho_i$ of atom $i$ shifted to the origin. In most cases, the atomic form factors $f_i(h)$ can be approximated by an expression consisting of several Gaussians, such as

$$f_i(h) = \left\{ \sum_{k=1}^{4} a_{ki} \exp \left[ -b_{ki} (\mathcal{F}^* h)^2/4 \right] + a_{0i} \right\} \times \exp \left[ -B_i (\mathcal{F}^* h)^2/4 \right].$$

(10)

The constant $B_i$ is the isotropic $B$ factor of atom $i$. The constants $a_{ki}$ and $b_{ki}$ can be obtained from *International Tables for X-ray Crystallography* (1974). The following derivations do not depend on the particular shape of the atomic form factors. In particular, the conclusions are valid for anisotropic form factors.

First derivatives of $F_{\text{calc}}(h)$ with respect to atomic positions can be derived from (9),

$$\frac{\partial F_{\text{calc}}(h)}{\partial \mathbf{r}_i} = 2\pi i f_i(h) \sum_{s=1}^{n_i} \mathbf{h} \cdot \partial_s \mathcal{F} \mathbf{1}$$

$$\times \exp \left[ 2\pi i \mathbf{h} \cdot (\partial \mathcal{F} \mathbf{r}_i + \mathbf{t}_i) \right]$$

(11)

where $\mathbf{1}$ is the identity vector.

**Fast Fourier transformation**

The computation of the exponential coefficients that occur in the direct summation formula [(9)] is expensive. An alternative to the direct summation method is to approximate (4) by discrete Fourier transformation of the electron density sampled at discrete grid points (Ten Eyck, 1977). A three-dimensional grid that covers the unit cell of the crystal is given by a set ($\Gamma$) of three-dimensional points

$$\Gamma = \{(a/N_a, b/N_b, c/N_c);$$

$$a = 0, \ldots, N_a - 1; b = 0, \ldots, N_b - 1;$$

$$c = 0, \ldots, N_c - 1\}.$$  

(12)

The points ($\mathbf{g} \in \Gamma$) are specified in fractional coordinates. The grid $\Gamma$ contains $N_a N_b N_c$ points.

Let the discrete Fourier transformation of $\rho(r)$ over the grid $\Gamma$ be defined as

$$\text{FT}(\rho; \mathbf{h}) = \sum_{\mathbf{g} \in \Gamma} \rho(\mathcal{F}^{-1} \mathbf{g}) \exp (2\pi i \mathbf{g} \cdot \mathbf{h})$$

(13)

where the sum ($\sum_{\mathbf{g} \in \Gamma}$) represents a three-dimensional sum over all grid points, i.e.,

$$\sum_{\mathbf{g} \in \Gamma} = \sum_{a=0}^{N_a-1} \sum_{b=0}^{N_b-1} \sum_{c=0}^{N_c-1}.$$  

(14)

Substitution of the continuous Fourier transformation in (4) by a discrete Fourier transformation over the finite grid $\Gamma$ yields an approximation for the structure factors,

$$F_{\text{calc}}(h) = \mathcal{N} \text{FT}(\rho; \Gamma; \mathbf{h})$$

(15)

where $\mathcal{N}$ is a normalization constant given by $|a||b||c|/(N_a N_b N_c)$. According to the Nyquist theorem (Brigham, 1974), (15) becomes exact when the electron density $\rho$ can be represented by a Fourier summation with coefficients not larger than $N_a/2$, $N_b/2$, $N_c/2$ in each dimension. Since the Gaussian form factors in (10) have an infinite Fourier spectrum, this implies that the accuracy of (15) is the better the finer the grid $\Gamma$ chosen. However, a very fine grid decreases the efficiency of the FFT method. At a given grid size the accuracy of (15) can be improved by artificially increasing the $B$ factor for each atom by a constant amount $B_0$ which is eliminated after Fourier transformation (Cochran, 1948; Ten Eyck, 1977). This broadens the electron density distribution and reduces the influence of the higher-order Fourier coefficients. For the choice of the coarseness of the grid and the artificial $B$-factor increase $B_0$, see Ten Eyck (1977). Typically, $B_0$ is set to 20-0 Å² and the grid is chosen approximately one third of the high-resolution limit of the data.

In principle, the electron density would have to be computed at all grid points specified in $\Gamma$. In the special case of the multiple Gaussian approximation (10), the electron density $\rho_i$ for atom $i$ is given by

$$\rho_i(r) = \sum_{k=1}^{4} a_{ki} \left( \frac{4\pi}{b_{ki} + B_i + B_0} \right)^{3/2} \exp \left[ \frac{-4\pi^2 (r - r_i)^2}{b_{ki} + B_i + B_0} \right]$$

(16)

which follows by Fourier transformation of the form factors $f_i$. Thus, this expression would have to be computed at each grid point and for each atom. This would actually require more calculations than the direct summation method. However, since the electron density of individual atoms falls off rapidly, it is a good approximation to compute only $\rho_i(r)$ in the neighborhood around the atoms, i.e.

$$\rho_i(r) = \widehat{\rho_i(r)} = \begin{cases} \rho_i(r), & \text{for } \mathcal{F} \mathbf{r} \in \Lambda_i \\ 0, & \text{otherwise}, \end{cases}$$

(17)

where the neighborhood $\Lambda_i$ is defined as the set of grid points $\mathbf{g}$ for which $\rho(\mathcal{F}^{-1} \mathbf{g})$ is greater than $10^{-7} \text{Å}^{-3}$. This results in spherical neighborhoods with radii of typically 3.0-6.0 Å depending on the atom type and $B$ factor. The artificial increase in the temperature factor ($B_0$) makes the neighborhoods of the atoms somewhat larger. The definition of $\Lambda_i$ is complicated by the fact that the neighborhood of an atom can penetrate the unit-cell boundaries. In this case, the selected grid points have to be projected back into the primary unit cell.
From (17) one obtains an approximation for $\rho^A$,
$$
\hat{\rho}^A(r) = \sum_{i=1}^{n}\rho_i(r).
$$
(18)

With (13) and (8) this yields an approximate expression for the structure factors,
$$
\overset{\leftrightarrow}{F}_{\text{calc}}(h) = N^0 \exp[B_0(\vec{\varphi}^*^0 h)^2/4] \sum_{i=1}^{n} \exp(2\pi i h \cdot t_i) \times \text{FT}(\hat{\rho}^A, \Gamma; \theta^0 \cdot h).
$$
(19)

The factor in front of the sum eliminates the artificial $B$-factor increase in (16). Inserting (19) into (1) one obtains an approximation for $\Delta$:
$$
\hat{\Delta} = \Delta(|F_{\text{obs}}|, \varphi_{\text{obs}}, \overset{\leftrightarrow}{F}_{\text{calc}}).
$$
(20)

Since the symmetry operators are applied after Fourier transformation, the electron density grid has to be computed only for the set of unique atoms. An alternative method would be to apply the symmetry operators in orthogonal coordinate space to $\rho^A$ on each grid point that has non-zero density.

## Factoring

The Fourier transformation in (19) can be obtained most efficiently by fast Fourier transformation (FFT) (Cooley & Tukey, 1965). The FFT memory requirement is of the order $N_a N_b N_c$. It is shown below that (19) can be factorized, thus reducing the memory requirements. This can be accomplished along similar lines to the reduction of a one-dimensional Fourier transformation into two one-dimensional Fourier transformations of half the size, which provides the basis for the one-dimensional FFT algorithm (Cooley & Tukey, 1965). The use of subgrids was also suggested by Raftery, Sawyer & Pawley (1985) in order to reduce memory requirements for their one-dimensional FFT algorithm to compute structure factors. It is shown below that the use of subgrids can be generalized to three dimensions.

A three-dimensional subgrid is defined by a set of points
$$
\Gamma'' = \{(a''/N_a'', b''/N_b'', c''/N_c'');
$$
$$a'' = 0, \ldots, N''_a - 1; b'' = 0, \ldots, N''_b - 1;
$$
$$c'' = 0, \ldots, N''_c - 1\}
$$
(21)

where $N''_a$, $N''_b$, $N''_c$, $N'_a$, $N'_b$, $N'_c$ are integer numbers such that
$$
N'_a N''_a = N_a, \quad N'_b N''_b = N_b, \quad N'_c N''_c = N_c,
$$
$$N'_a < N_a, \quad N'_b < N_b, \quad N'_c < N_c, \quad N''_a \leq N_a, \quad N''_b \leq N_b, \quad N''_c \leq N_c.
$$
(22)

The points of $\Gamma''$ and a set of points $\Gamma'$ defined by
$$
\Gamma' = \{(a'/N_a, b'/N_b, c'/N_c);
$$
$$a' = 0, \ldots, N'_a - 1; b' = 0, \ldots, N'_b - 1;
$$
$$c' = 0, \ldots, N'_c - 1\}
$$
(23)
The points in $\Gamma'$ provide the offset that is added to the subgrid $\Gamma''$. We can then formally write
$$
\Gamma = \Gamma'' + \Gamma'
$$
(24)

where the '+' operator adds all points of set $\Gamma'$ to all points of set $\Gamma''$. Fig. 1 illustrates the factoring of $\Gamma$ for a special two-dimensional case.

With these definitions one can rewrite (13),
$$
\text{FT}(\rho, \Gamma; h) = \sum_{g' \in \Gamma'} \sum_{g'' \in \Gamma''} \exp(2\pi i (g' + g'') \cdot h) \times \rho[\vec{\varphi}^{-1}(g' + g'')].
$$
(25)

By making use of the relation reaction $\exp(a+b) = \exp(a) \exp(b)$ one may factorize this equation, i.e.
$$
\text{FT}(\rho, \Gamma; h) = \sum_{g' \in \Gamma'} \exp(2\pi i g' \cdot h) \text{FT}(\rho, \Gamma'' + g'; h \mod \Gamma'').
$$
(26)

The use of $h \mod \Gamma''$ rather than $h$ in (26) reflects the fact that the projection of the reflection indices into the periodic subgrid $\Gamma''$ does not change the value of $\exp(2\pi i g' \cdot h)$. The mod function operates on each component $h$, $k$, $l$ of the vector $h$ separately, i.e. we define
$$
h \mod \Gamma'' = \begin{cases} 
    h \mod N''_a \\
    k \mod N''_b \\
    l \mod N''_c
\end{cases}
$$
(28)

Fig. 1. Illustration of the factoring of the grid $\Gamma$ (small dots) into the subgrids $\Gamma''$ (crosses) and $\Gamma'$ (circles) for a two-dimensional case. The vectors $a$ and $b$ are unit-cell vectors. The grid point $g$ can be represented as $g' + g''$. In this example, the Fourier transformation over the grid $\Gamma$ is a sum of 12 Fourier transformations over the subgrid $\Gamma''$ where each subgrid is translated by a vector $g' \in \Gamma'$. 

where \( x \mod y \) is defined as the remainder of the integer division of \( x \) by \( y \). This projection of the reflection indices \( \mathbf{h} \) into the subgrid \( I'' \) is necessary since the result of the subgrid Fourier transform \([27]\) will only be available for \( \mathbf{h} \in I'' \). By inserting (26) into (19) we obtain for the calculated structure factors the expression

\[
F_{\text{calc}}(\mathbf{h}) = \mathcal{N} \exp \left[ B_0 (\mathcal{F} \mathbf{r}^* \mathbf{h})^2 / 4 \right] 
\times \sum_{\mathbf{g} \in I''} n_\mathbf{g} \exp \left[ 2\pi i (\mathbf{g} \cdot \mathcal{O}_\mathbf{g}^* \mathbf{h} + t \cdot \mathbf{h}) \right] 
\times \text{FT} \left( \mathcal{F} (\mathbf{p}_\mathbf{A}, \mathcal{F}^* \mathbf{h} \mod I'') \right). \tag{29}
\]

The summation over \( I'' \) has to be carried out for the number \( |\Omega| \) of the set of observed reflections \( \Omega \). This requires memory of the order of \( |\Omega| \). The Fourier transformations of \( \mathcal{F} \mathbf{r}^* \) require memory of the order of \( N_\mathbf{g}^* N_\mathbf{g}^t N_\mathbf{g}^r \). Thus, the total memory requirement for (29) is of the order of \( N_\mathbf{g}^* N_\mathbf{g}^t N_\mathbf{g}^r + |\Omega| \). The potential drawback of (29) appears to be the computation of the exponential coefficients. Unless additional memory is used to precompute and store these coefficients, they have to be recomputed each time the structure factors are required. Equation (29) provides the basis of the SGFFT algorithm.

**First derivatives**

In this section we will show how to evaluate the first derivatives of \( \hat{\Delta} \) \([20]\) with respect to all model parameters \( p_i \). As was pointed out by Cochran \( [1948] \), the chain rule can be used to compute the derivatives of the crystallographic residual. However, since \( \hat{\Delta} \) is an approximation of \( \Delta \), the derivatives of \( \hat{\Delta} \) will then be approximations of the derivatives of \( \Delta \). The following derivation generalizes the work of Lunin & Urzhumtsev \( [1985] \) to the SGFFT algorithm introduced in the previous section.

Using the chain rule one can write

\[
\frac{\partial \hat{\Delta}}{\partial p_i} = \sum_{\mathbf{h} \in \Omega} \frac{\partial \hat{\Delta}}{\partial F_{\text{calc}}(\mathbf{h})} \frac{\partial F_{\text{calc}}(\mathbf{h})}{\partial p_i} + \frac{\partial \hat{\Delta}}{\partial F_{\text{calc}}(\mathbf{h})} \frac{\partial F_{\text{calc}}(\mathbf{h})}{\partial p_i}, \tag{30}
\]

where \( F_{\text{calc}}(\mathbf{h}) \) and \( F_{\text{calc}}(\mathbf{h})^* \) denote the real and the imaginary part of \( F_{\text{calc}}(\mathbf{h}) \), respectively. From (29) it follows that

\[
\frac{\partial F_{\text{calc}}(\mathbf{h})}{\partial p_i} = \mathcal{N} \exp \left[ B_0 (\mathcal{F} \mathbf{r}^* \mathbf{h})^2 / 4 \right] 
\times \sum_{\mathbf{g} \in I''} n_\mathbf{g} \exp \left[ 2\pi i (\mathbf{g} \cdot \mathcal{O}_\mathbf{g}^* \mathbf{h} + t \cdot \mathbf{h}) \right] 
\times \text{FT} \left( \mathcal{F} (\mathbf{p}_\mathbf{A}, \mathcal{F}^* \mathbf{h} \mod I'') \right) \tag{31}
\]

where we define

\[
\frac{\partial F_{\text{calc}}(\mathbf{h})}{\partial p_i} = \frac{\partial F_{\text{calc}}(\mathbf{h})}{\partial p_i} + i \frac{\partial F_{\text{calc}}(\mathbf{h})}{\partial p_i}. \tag{32}
\]

In principle one could compute the derivatives by inserting (31) into (30), but this would be impractical since it would require a Fourier transformation for each parameter \( p_i \) of each atom \( i \). It is shown in the following that instead of carrying out the Fourier transformation of \( \partial \hat{\Delta} / \partial p_i \), one can obtain an expression that involves a Fourier transformation of coefficients that are related to \( \partial \hat{\Delta} / \partial F_{\text{calc}} \). Let us define

\[
\frac{\partial \hat{\Delta}}{\partial F_{\text{calc}}(\mathbf{h})} = \frac{\partial \hat{\Delta}}{\partial F_{\text{calc}}(\mathbf{h})} + i \frac{\partial \hat{\Delta}}{\partial F_{\text{calc}}(\mathbf{h})}, \tag{33}
\]

\( i.e. \partial \hat{\Delta} / \partial F_{\text{calc}}(\mathbf{h}) \) is a complex number whose real part contains the first derivative of \( \hat{\Delta} \) with respect to the real part of the structure factor \( F_{\text{calc}}(\mathbf{h}) \) and whose imaginary part contains the first derivative of \( \hat{\Delta} \) with respect to the imaginary part of the structure factor \( F_{\text{calc}}(\mathbf{h}) \).

If we make the reasonable assumption that \( \hat{\Delta} \) is a real function, it follows from (30) that

\[
\frac{\partial \hat{\Delta}}{\partial p_i} = \text{Re} \left\{ \sum_{\mathbf{h} \in \Omega} \left( \frac{\partial \hat{\Delta}}{\partial F_{\text{calc}}(\mathbf{h})} \right) \ast \frac{\partial F_{\text{calc}}(\mathbf{h})}{\partial p_i} \right\}, \tag{34}
\]

where \( \text{Re} \) denotes the real part of its argument, and \( \ast \) the complex conjugate. The Fourier transformation in (31) involves a sum over all points \( \mathbf{g} \in I'' \). The partial derivatives of \( \mathcal{F} \mathbf{r}^* \) are zero for grid points outside the neighborhood \( \Lambda_i \) of atom \( i \),

\[
\frac{\partial \mathcal{F} \mathbf{r}^*}{\partial p_i} (\mathbf{g}') = \sum_{j=1}^{n_\mathbf{g}^t} \frac{\partial \mathcal{F} \mathbf{r}^*}{\partial \mathcal{F} \mathbf{r}^* (\mathbf{g})} (\mathbf{g}') = \frac{\partial \mathcal{F} \mathbf{r}^*}{\partial \mathcal{F} \mathbf{r}^* (\mathbf{g})} (\mathbf{g}')
\]

\[
= \left\{ \begin{array}{ll}
\frac{\partial \mathcal{F} \mathbf{r}^*}{\partial \mathcal{F} \mathbf{r}^* (\mathbf{g})} & \mathbf{g} \in \Lambda_i, \\
0 & \text{otherwise}
\end{array} \right. \tag{35}
\]

which follows from (6) and (17). Therefore, it is more efficient to carry out the summation over \( \Lambda_i \) explicitly rather than by FFT's.

We will show below that the sum over \( \mathbf{h} \) in (30) can be replaced by a FFT which uses indices \( \mathbf{h}' \) that are related to \( \mathbf{h} \). The Fourier transformation in (31) involves coefficients of the form

\[
\exp \left[ 2\pi i \mathbf{g}' \cdot (\mathcal{O}_\mathbf{g}^* \mathbf{h} \mod I'') \right] \tag{36}
\]

where \( \mathbf{g}' \in I'' + \mathbf{g}' \). Since the summation over \( \mathbf{g}' \) is carried out explicitly this leaves

\[
\mathbf{h}' = \mathcal{O}_\mathbf{g}^* \mathbf{h} \mod I'' \tag{37}
\]

as a candidate for Fourier summation. Let us define a set of indices for each \( \mathbf{h}' \in I'' \) and each symmetry operator,

\[
\text{map}(\mathbf{h}', \mathcal{O}_\mathbf{g}; I'') = \{ \mathbf{h} \in \Omega \ (and \ \mathcal{O}_\mathbf{g}^* \mathbf{h} \mod I'' = \mathbf{h}') \}. \tag{38}
\]

where the mod function has been defined in (28). Let \( B(\mathbf{h}') \) be complex numbers that are obtained from the first derivatives of \( \hat{\Delta} \) after summation over the sets map(\( \mathbf{h}', \mathcal{O}_\mathbf{g}; I'' \)) and application of the symmetry
operators,

\[
B(h^*) = \sum_{s=1}^{n} \sum_{h \in \text{map}(h^*, \mathcal{O}_s, \Omega)} \exp[2\pi i (g' \cdot \mathcal{O}_s^* \cdot h + t_s \cdot h)] \\
\times \left[ \frac{\partial \Delta}{\partial F_{\text{calc}}(h)} \right]^* 
\]

(39)

for all \( h^* \in \Omega^* \). The expression in square brackets was defined in (33) above, the * means the complex conjugate. Note that the second sum is defined as zero if the set map \((h^*, \mathcal{O}_s; \Omega^*)\) is empty. Another way of expressing \( B(h^*) \) is to generate for each observed reflection \( h \) the set of symmetry-related reflections. The resulting reflection indices are projected into \( \Omega^* \). For each projected index the complex conjugate of the first derivative of \( \Delta \) with respect to \( F_{\text{calc}}(h) \) is then multiplied by the exponential coefficient as given in (39) and the result is accumulated in \( B(h^*) \). With these definitions one obtains by resummation of (31) and (34)

\[
\frac{\partial \Delta}{\partial p_i} = \operatorname{Re} \left\{ \sum_{g' \in \Omega^*} \sum_{g \in \Omega^*} \hat{\Delta} \left[ F_{\text{calc}}^{-1}(g' + g') \right] \right\} \\
\times \text{FT}(B, \Omega^*; g^*)
\]

(40)

where the Fourier transform is applied to \( B(h^*) \). Equation (40) allows computation of all partial derivatives using the same number of Fourier transformations needed to compute the structure factors [(29)]. The CPU time and memory requirements for (40) are identical to the ones for (29).

### Incorporation of Hermitian symmetry into the SGFFT algorithm

Hermitian symmetry of the structure factors \([F_{\text{calc}}(h) = F_{\text{calc}}(-h)]\) can easily be incorporated into (29) and (40). The number of operations required for the Fourier transformations in (29) and (40) is then reduced by a factor of two. The set \( \Omega \) can be reduced to one hemisphere, i.e. without restriction the hemisphere defined by \( h > 0 \). However, the Fourier transformation of \( B \) in (40) includes both hemispheres. This implies that (40) has to be multiplied by a factor \( 1/2 \). The right-hand side of (40) will have a zero imaginary part because of the Hermitian symmetry.

### Vectorization of the SGFFT algorithm

Fig. 2 shows a flowchart of the SGFFT algorithm described in (29) and (40). All operations are highly vectorizable with the exception of the computation of the electron density grid. The vectorization was accomplished automatically by the Cray and Convex compilers. Machine-dependent programming was not necessary. The electron density calculation is only moderately vectorizable since the use of look-up tables to compute the form factors [(10)] and the organization of the neighborhoods \( A_i \) of the atoms produce memory conflicts. Thus, at the present state of the program, the electron density calculation is memory rather than CPU bound. It is important to note that the reduction of Gaussian factors in (10) to three (as is used by Agarwal, 1978) reduces the CPU time required to compute the electron density on a supercomputer by less than 20% (not shown).

Efficient library routines were used for the three-dimensional FFTs. The algorithm in Fig. 2 is also suitable for parallelization, so that the algorithm should perform well on a multiprocessor machine with shared memory. Largely parallel machines would in fact be better suited for the electron density calculation.

The SGFFT algorithm is part of the X-PLOR program (Brünger, Karplus & Petsko, 1989; Brünger, 1988a, b). The program X-PLOR is available on request from ATB.

---

Fig. 2. Flowchart showing the main stages of the algorithm described in this work to compute the structure factors, the target function, and the partial derivatives with respect to the atomic parameters \( p_i \). Blocks enclosed by shaded lines represent (29) and (40) which require large memory.
Table 1. CPU times (s) for computation of the target function $\Delta$ and partial derivatives with respect to atomic positions

<table>
<thead>
<tr>
<th>Task</th>
<th>Vax 8700*</th>
<th>Convex-C1†</th>
<th>Cray-XMP‡</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A) Crambin§</td>
<td>P2$_1$, 1·5.8-8.0 Å, 327 non-H atoms, 5576 reflections,</td>
<td>$</td>
<td>I'</td>
</tr>
<tr>
<td></td>
<td>Direct summation</td>
<td>440·5</td>
<td>72·55</td>
</tr>
<tr>
<td></td>
<td>$\rho^A$ (unique atoms)</td>
<td>11·5</td>
<td>11·52</td>
</tr>
<tr>
<td></td>
<td>$P_1$-FFT</td>
<td>15·7</td>
<td>2·24</td>
</tr>
<tr>
<td></td>
<td>Symmetry, factoring</td>
<td>1·1</td>
<td>0·70</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>28·3</td>
<td>14·5</td>
</tr>
<tr>
<td>(B) Aspartate aminotransferase¶</td>
<td>C222.2, 2·8-8-8.0 Å, 3086 non-H atoms, 8124 reflections,</td>
<td>$</td>
<td>I'</td>
</tr>
<tr>
<td></td>
<td>Direct summation</td>
<td>10·355</td>
<td>1572</td>
</tr>
<tr>
<td></td>
<td>$\rho^A$ (unique atoms)</td>
<td>56</td>
<td>41·3</td>
</tr>
<tr>
<td></td>
<td>$P_1$-FFT</td>
<td>372·2</td>
<td>21·8</td>
</tr>
<tr>
<td></td>
<td>Symmetry, factoring</td>
<td>34</td>
<td>8·1</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>463·2</td>
<td>71·2</td>
</tr>
<tr>
<td>(C) Alkaline phosphatase**</td>
<td>I222, 2·4-8-8.0 Å, 6529 non-H atoms, 38264 reflections,</td>
<td>$</td>
<td>I'</td>
</tr>
<tr>
<td></td>
<td>Direct summation</td>
<td>76·920</td>
<td>11654</td>
</tr>
<tr>
<td></td>
<td>$\rho^A$ (unique atoms)</td>
<td>170</td>
<td>135·4</td>
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<tr>
<td></td>
<td>$P_1$-FFT</td>
<td>1046</td>
<td>63·0</td>
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<td></td>
<td>Symmetry, factoring</td>
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<td></td>
<td>Total</td>
<td>1423</td>
<td>251·9</td>
</tr>
<tr>
<td></td>
<td>PROFFT†</td>
<td>602</td>
<td>—</td>
</tr>
</tbody>
</table>

† Using not more than 16 Mbyte memory, 64-bit precision. Using VECLIB (Convex Computer Corp., 1983) Z3DFFT routine for 3D FFT.  
‡ Using an [81, 40, 48] grid and an identical subgrid.  
§ Using an [180, 96, 96] grid and a [180, 96, 24] subgrid. Owing to systematic absences only half the symmetry operators of space group C222₁ have to be applied to compute the structure factors $\langle x, y, z \rangle$, $\langle -x, y, z \rangle$, $\langle x, -y, z \rangle$, $\langle x, y, -z \rangle$.  
¶ Using a [240, 216, 96] grid and a [240, 216, 16] subgrid. Owing to systematic absences only half the symmetry operators of space group I222 have to be used to compute the structure factors $\langle x, y, z \rangle$, $\langle -x, y, z \rangle$, $\langle x, -y, -z \rangle$, $\langle x, -y, z \rangle$.  
** Using a [240, 216, 96] grid and a [240, 216, 16] subgrid. Owing to systematic absences only half the symmetry operators of space group I222 have to be used to compute the structure factors $\langle x, y, z \rangle$, $\langle -x, y, z \rangle$, $\langle x, -y, -z \rangle$, $\langle x, -y, z \rangle$.

The application of the factoring and symmetry operators appears to be almost negligible on the Vax 8700 computer. Even in the case of alkaline phosphatase with a subgrid $I''$ a sixth of the size of the original grid $I$, the operations take not more than a fifth of the time spent for the execution of the FFT’s. On the supercomputers, this situation is not quite as favorable. This is mainly because the FFT’s are library routines that make use of the particular architecture of the machine, whereas the routine that carried out the factoring and symmetry operations was written in Fortran. Nevertheless, the degree of vectorizability for this routine is high; for example, in the case of alkaline phosphatase the routine is 47 times faster on the Cray-XMP than on the Vax 8700. Nonvectorizable code executes approximately six times faster on the Cray-XMP than on the Vax 8700.

The direct summation method [(9)] appears to be highly vectorizable. This is reflected by the CPU times in Table 1; e.g. the CPU time for the direct summation is increased by a factor 100 between the Cray-XMP and the Vax 8700. The SGFFT algorithm is more complicated than the direct summation method. Despite this, it has been shown that it is possible to vectorize the SGFFT algorithm efficiently.

### Results

Table 1 clearly shows that the SGFFT algorithm is up to two orders of magnitude faster than the direct summation method. The SGFFT algorithm appears to perform best for the larger systems, aspartate aminotransferase and alkaline phosphatase. On the Vax 8700 computer, the FFT’s require most of the total CPU time whereas on both the superminicomputer (Convex-C1) and the supercomputer (Cray-XMP) the electron density calculation becomes the most expensive part of the computation. This clearly shows that, at the present state of the program, the introduction of space-group-specific FFT’s could reduce the total CPU time for the SGFFT method by not more than a factor of two.
vectorized SGFFT algorithm is up to 26 times faster than the vectorized direct summation method on the Cray-XMP. Since the SGFFT algorithm is designed in a general and simple way, it is expected that it could be efficiently implemented on supercomputers with new parallel architectures.

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References


Crystallographic Refinement by Simulated Annealing: Application to Crambin

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Abstract

A detailed description of the method of crystallographic refinement by simulated annealing is presented. To test the method, it has been applied to a 1.5 Å resolution X-ray structure of crambin. The dependence of the success of the simulated annealing protocol with respect to the temperature of the heating stage is discussed. Optimal success is achieved at relatively high temperatures. Regardless of the protocol used, the molecular-dynamics refined structure always yields an improved R factor compared with restrained least-squares refinement without manual re-fitting. The differences between the various refined structures and the corresponding electron density maps are discussed.

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